Journal of Thermal Analysis, Vol. 34 (1988) 871–873

ON TIME-DEPENDENT HEAT CONDUCTION IN LAYERED MATERIALS

Towards solving the infinite series problem

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(Received May 29, 1987)

The asymptotic behaviour of the temperature decay constants in Fourier's series is shown to be a powerful tool for a simplification of the latter, which may considerably save computing time, simplify the alternative series obtained from the Laplace transformation, thus, solve, at least in principle, the convergence problem of both standard series and lift the problem of matching both ones.

The time-dependent heat conduction in layered media faces one with the (an)harmonic Fourier analysis, i.e. with infinite series of decay terms [1, 2]. These series converge ill either for large, or for short times. But good convergence is necessary to recognize individual layers in the temperature or effusivity curve [3–5]. Thus, for practical purposes, it is highly desirable to get a systematic evaluation of the cut-off time, below (above) which the other series becomes preferable, or, alternatively, to improve the convergence of the known series. The goal of this note is to report on the first step in this direction. The basic idea consists in transforming the Fourier series into a lattice sum by using the asymptotics of the temporal decay constants, τ_n, reported recently [5] and then performing the theta function transformation (which is otherwise only applicable, when the Fourier series is harmonic).

Let us assume, that the heat conduction is quasi-onedimensional and that the temperature decay at a given point after Dirac-pulse excitation is described by the Fourier series

$$\theta(t) = \sum_{n=0}^{\infty} F(\tau_n)e^{-t/\tau_n} \tag{1}$$

where the prefactors $F(\tau_n)$ are determined by the experimental and boundary conditions. For N homogeneous layers of thicknesses L_i with (temperature-independent) diffusivities $\varkappa_i$ ($i = \overline{1, N}$) holds [5]

$$\tau_n \to (\eta/n\pi)^2 \quad \text{as} \quad n \to \infty, \tag{2}$$

where

$$\eta = \sum_{i=1}^{N} \eta_i = \sum_{i=1}^{N} L_i \Big/ \sqrt{\varkappa_i}. \tag{3}$$

The convergence of the τ_n's to their asymptotic value depends upon the case under consideration; moreover, one must carefully investigate the corresponding convergence of the prefactors in (1) [5].

Let $\tau_{\bar{n}}$ be the first (largest) decay constant, for which (2) is applicable. Then (1) can be rewritten as

$$\theta(t) = \theta_1(t) + \theta_2(t), \tag{4a}$$

$$\theta_1(t) = \sum_{n=0}^{\infty} F_n e^{-n^2\pi^2 t/\eta^2}, \quad F_n = F((\eta/n\pi)^2), \tag{4b}$$

$$\theta_2(t) = \sum_{n=0}^{\bar{n}-1} (F(\tau_n) e^{-t/\tau_n} - F_n e^{-n^2\pi^2 t/\eta^2}). \tag{4c}$$

Therefore, the number of τ_n's to be calculated from the transcendent eigenvalue equation may considerably decrease [5]. Moreover, θ_1 represents a lattice sum, for which we can utilize the theta function transformation [6].

$$H(m) = \sum_{n=-\infty}^{+\infty} F_{n-m} e^{-(n-m)^2 b} \tag{5}$$

is a periodic function of m with unit period. Thus, it can be represented as a Fourier series,

$$H(m) = \sum_{g=-\infty}^{+\infty} H_g e^{2\pi i g m} \tag{6}$$

with

$$H_g = \int_0^1 H(m) e^{-2\pi i g m} dm = \int_{-\infty}^{+\infty} F_m e^{-m^2 b - 2\pi i g m} \mathrm{d}m \tag{7}$$

by insertion of (5). Therefore,

$$\sum_{n=-\infty}^{+\infty} F_n e^{-n^2 b} = \sum_{g=-\infty}^{+\infty} \int_{-\infty}^{+\infty} F_m e^{-m^2 b - 2\pi i g m} \mathrm{d}m. \tag{8}$$

This is the desired relation in general form (for its generalization to more dimensions, cf. [6]).

Let us illustrate (8) by two simple examples. (i), $F_n = F = \text{const}$,

$$\sum_{n=-\infty}^{+\infty} e^{-n^2 b} = \sum_{g=-\infty}^{+\infty} \sqrt{\frac{\pi}{b}}\, e^{-\pi^2 g^2/b}; \tag{9}$$

(ii), $F_n = F\cos(2\pi\omega n)$,

$$\sum_{n=-\infty}^{+\infty} \cos(2\pi\omega n) e^{-n^2 b} = \frac{1}{2}\sqrt{\frac{\pi}{b}} \sum_{g=-\infty}^{+\infty} \left(e^{-\pi^2(g-\omega)^2/b} + e^{-\pi^2(g+\pi)^2/b}\right). \tag{10}$$

Consequently, the asymptotic form allows for a systematic and very simple form of the terms of the series, alternative to (1), for the case of anharmonic Fourier analysis [1], too (in fact, the anharmonic case becomes asymptotically harmonic, cf. [7]). In contrast, the expansion of the Laplace transform of (1) with respect to terms like $\exp(-c^2/t)$ contains increasingly complicated combinations of the η_i's.

When now η^2 is much larger, than the times of observation after excitation (e.g., when the stack contains a thick layer or substrate, cf. (3)), (8) reduces to one (!) or few terms only. Hence, the number of terms to be used at all reduces to the order of $\bar{n}$, what may be much less, than the terms needed from the standard series.

In summary, the calculation of the temperature decay according to (4) solves, at least in principle, the convergence problem of the standard series, reduces considerably the computer time, and lifts the problem of matching the alternative standard series (cf. [5]).

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